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Cyclic public goods games: Compensated coexistence among mutual cheaters stabilized by optimized penalty taxation

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We study the problem of stabilized coexistence in a three-species public goods game in which each species simultaneously contributes to one public good while freeloading off another public good ("cheating"). The proportional population growth is governed by an appropriately modified replicator equation, depending on the returns from the public goods and the cost. We show that the replicator dynamic has at most one interior unstable fixed point and that the population becomes dominated by a single species. We then show that by applying an externally imposed penalty, or "tax" on success can stabilize the interior fixed point, allowing for the symbiotic coexistence of all species. We show that the interior fixed point is the point of globally minimal total population growth in both the taxed and untaxed cases. We then formulate an optimal taxation problem and show that it admits a quasilinearization, resulting in novel necessary conditions for the optimal control. In particular, the optimal control problem governing the tax rate must solve a certain second-order ordinary differential equation.

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I. INTRODUCTION

The public goods game is a mathematical representation of the Tragedy of the Commons [1,2]—the idea that in situations of a shared common product such as clean surroundings or common grazing area, there is less incentive to contribute than there is to "cheat," or freeload on the contributions of others. In Lloyd's original 1833 lecture, the commons refers to the common grazing land held in England at the time [1]. Hardin later related this problem to nuclear proliferation during the 1960s [2]. Within the public goods game, cheating is a more profitable choice than producing; in this way, it is intellectually similar to the prisoner's dilemma (see, e.g., [3,4]), and various approaches to resolving the tragedy have been taken (see, e.g., [5]).

However, the division of labor that led to the development of advanced human societies was based on the idea of specialization combined with an exchange of produced goods; this could be thought of as *mutual freeloading*. In this scenario it is essential that there are multiple, complementary products made by different members of the society. Although such products, such as making bread or raising livestock, are easily exchanged or withheld and therefore not at all public goods, it seems relevant to consider a situation with multiple public goods. The mathematical model we propose here treats the situation in which there are three choices, each of which involves producing one good while simultaneously freeloading on another.

The public goods game [6] poses the following dilemma to a group of N agents: Each agent is asked to contribute c monetary units to a public good. If the agent contributes, the contribution earns a rate of return r providing rc monetary units to the public. The rc monetary units are then shared among the population of N agents. Thus, if k individuals contribute, a contributing individual (cooperator) earns $rck/N - c$ monetary units, while a noncontributing individual (freeloader) earns rck/N monetary units. Rational agents would therefore choose not to contribute as long as $r < N$, which is generally assumed.

Direct translation of this dynamic into differential equations frequently takes the following form: two fitness functions f_C for cooperators and f_D for freeloaders are defined. Population growth (proportional or actual) is then linked to these fitness functions. The direct translation of this dynamic into differential equations (usually) leads to cooperator population collapse (because contribution to the public good is less beneficial than freeloading). This is illustrated in, e.g., Ref. [7]. Recently models including synergistic effects that capture the fact that it is more attractive to participate than to freeload have been proposed [8]. Reference [9] provides an excellent review of differential equation models of public goods.

In [7,10,11], a three-population model is considered that includes cooperators, freeloaders, and a *vacancy* population that models free space within the system; birth can only occur if space is available. The authors show that a Hopf bifurcation occurs and a limit cycle emerges within the cooperator and freeloader populations. In [12] the authors study diversified contributions from a finite population and the emergence of cooperative behavior therein. Archetti and Scheuring study the emergence of mixed populations of cooperators and freeloaders in dynamic public goods games and evaluate these in the context of the prisoner's dilemma and the volunteer's dilemma [13].

Additional work has been done on the emergence of cooperation in public goods games when exogenous influences are present. In [14], the authors consider a finite population involved in a public goods game where punishment for freeloading is possible, but agents may abstain from contributing

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or receiving any benefits. In [15], Hauert considers reputation as a kind of reward-punishment system within the public goods context; Cressman *et al.* [16] discusses a dynamic public goods game with institutional incentives, which are similar to the tax structures we consider in this paper.

Finally, much work has been done on the incorporation of spatial dynamics into the dynamic public goods problem. Wakano [17] studied freeloading in the context of a farming scenario, showing how cooperation is influenced to emerge as a result of the spatial component of the model. In [18], the authors show that cooperation within an ecological public goods model can be promoted by differing pattern formation processes. This work is continued in [19], where the authors demonstrate the creation of spatial chaos in spatial public goods games. Finally, work in nonhomogenous spatial settings (i.e., scale-free graphs) is studied in [20]. In particular, this work also included an exogenous investment element, similar to [16].

In this paper, we propose a variation of the public goods game with three species, each contributing to one public good while freeloading off another public good, creating a three-way symbiotic relationship. For the case of three such mutual producer-freeloader pairs, we find that the standard replicator model leads to the complete dominance of one population and that coexistence among the mixed population of three species is unstable. We also show that this coexistence equilibrium point corresponds to minimum population growth. We then show that the inclusion of a penalty or "tax" can stabilize the mixed population equilibrium. We prove that the interior mixed population is the unique mixed-species fixed point and show that the dynamics we study cannot exhibit isolated periodic orbits. In particular, we show that the three-species public goods game is diffeomorphic to rock-paper-scissors under the evolutionary game theoretic replicator dynamic. We then examine the problem of designing an optimal success tax and formulate this problem as an optimal control problem. We show that a certain quasilinearization of the problem admits a second order ordinary differential equation that the quasilinear optimal control must satisfy.

The remainder of this paper is formulated as follows: In Sec. II we discuss notational preliminaries. In Sec. III we formulate the three-species cyclic public goods game and derive results, including stability. We then present stability results for a three-species public goods game with penalty tax in Sec. IV. We discuss the optimal control problem for the taxation in Sec. V, followed by conclusions and future directions in Sec. VI.

II. NOTATION AND PRELIMINARIES

A. Evolutionary formulation

Throughout this paper, we use $\mathbf{x} = \langle x_1, \ldots, x_n \rangle$ to denote a column vector in $\mathbb{R}^n$. Let $\Delta_n \subset \mathbb{R}^n$ be the standard n-dimensional simplex defined by

$$\Delta_n = \left\{ \mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 1 \text{ and } 0 \leqslant x_i \leqslant 1 \right\}. \quad (1)$$

Assume we have a population of $n \geqslant 1$ species, so that U_i is the (raw) number of individuals of species i, and the total population size is $M = \sum_i U_i$. For each $i = 1, \ldots, n$, let $u_i = U_i/M$ denote the proportion (frequency) of population i. Suppose the growth dynamics of each population is governed by the differential equation

$$\dot{U}_i = U_i f_i(\mathbf{u}), \quad (2)$$

where $\mathbf{u} = \langle u_1, \ldots, u_n \rangle$ is the vector of population proportions and $f_i : \mathbb{R}^n \to \mathbb{R}$ is continuous and differentiable on Δ_n for $i = 1, \ldots, n$. Elementary calculus (the quotient rule) shows that if

$$\bar{f} = \sum_{i=1}^{n} u_i f_i(\mathbf{u}), \quad (3)$$

then for each $i = 1, \ldots, n$

$$\dot{u}_i = u_i [f_i(\mathbf{u}) - \bar{f}(\mathbf{u})], \quad (4)$$

which is the replicator dynamic [21–23]. Moreover, the growth of the entire population is given by

$$\dot{M} = M \bar{f}, \quad (5)$$

where the variable growth-rate function $\bar{f}$ is given by Eq. (3).

If there is some (payoff) matrix $\mathbf{A}$ so that

$$f_i(\mathbf{u}) = \mathbf{e}_i^T \mathbf{A} \mathbf{u}, \quad (6)$$

where $\mathbf{e}_i$ is the ith standard basis vector in n-dimensional Euclidean space, then Eq. (4) can be rewritten as

$$\dot{u}_i = u_i \left(\mathbf{e}_i^T \mathbf{A} \mathbf{u} - \mathbf{u}^T \mathbf{A} \mathbf{u} \right). \quad (7)$$

See [24] for a discussion of alternative evolution equations.

B. Rock-paper-scissors games

Following [21], the generalized rock-paper-scissors game is described by the payoff matrix

$$\mathbf{A}_{\text{rps}} = \begin{bmatrix} 0 & -a_2 & b_3 \\ b_1 & 0 & -a_3 \\ -a_1 & b_2 & 0 \end{bmatrix}, \quad (8)$$

where $a_1, a_2, a_3, b_1, b_2, b_3 > 0$. Here the columns (and rows) correspond to the strategies *rock*, *paper*, and *scissors* (RPS) in order. From Theorem 8 of [25] we state the following:

Lemma II.1. There is a unique interior fixed point $\mathbf{u}^* \in \Delta_3$ for Eq. (7) with $\mathbf{A}_{\text{rps}}$ and the following are equivalent:

(1) $\mathbf{u}^*$ is asymptotically stable,

(2) $\mathbf{u}^*$ is globally stable,

(3) $\det \mathbf{A}_{\text{rps}} > 0$, and

(4) $\mathbf{u}^{*T} \mathbf{A}_{\text{rps}} \mathbf{u}^* > 0$.

Further, if $\det \mathbf{A}_{\text{rps}} = 0$, then $\mathbf{u}^*$ is a nonlinear center, and if $\det \mathbf{A}_{\text{rps}} < 0$, then $\mathbf{u}$ is asymptotically unstable and all orbits tend to the boundary of Δ_3.

Corollary II.2 (Ref. [21], p. 483). There are no isolated orbits for Eq. (7) with $\mathbf{A}_{\text{rps}}$, i.e., RPS does not admit a limit cycle.

It is worth noting that adding constants to the columns of $\mathbf{A}_{\text{rps}}$ will not change the qualitative behavior of the orbits of Eq. (7) [22] (or see Sec. 2.4 of [21]); i.e., the orbits of the resulting solutions will be diffeomorphic to those of the RPS system. Lemma II.1 was originally established in [25] and [26]. The results are succinctly reported in [21].

III. THREE-SPECIES PUBLIC GOODS

We propose the following three-species public goods game with populations $U(t)$, $V(t)$, and $W(t)$. Assume individuals in the first species cooperate among themselves by contributing to public good A but are freeloaders on public good C, which is produced by species 3. Similarly, individuals in the second species cooperate by contributing to public good B but are freeloaders of public good A, while individuals in species 3 cooperate to produce to public good C but freeload on public good B.

The resulting population dynamics are as follows:

$$\dot{U} = U\left(\frac{c_A r_A U}{U+V+W} - c_A + \frac{c_C r_C W}{U+V+W}\right), \tag{9}$$

$$\dot{V} = V\left(\frac{c_B r_B V}{U+V+W} - c_B + \frac{c_A r_A U}{U+V+W}\right), \tag{10}$$

$$\dot{W} = W\left(\frac{c_C r_C W}{U+V+W} - c_C + \frac{c_B r_B V}{U+V+W}\right), \tag{11}$$

where r_A,r_B,r_C and c_A,c_B,c_C are the rates of return and costs, respectively, for the three public goods, and we assume $r_A,r_B,r_C > 1$. Define

$$p_A(u,v,w) = c_A r_A u - c_A + c_C r_C w, \tag{12}$$

$$p_B(u,v,w) = c_B r_B v - c_B + c_A r_A u, \tag{13}$$

$$p_C(u,v,w) = c_C r_C w - c_C + c_B r_B v. \tag{14}$$

Then

$$\dot{U} = U p_A(u,v,w), \tag{15}$$

$$\dot{V} = V p_B(u,v,w), \tag{16}$$

$$\dot{W} = W p_C(u,v,w), \tag{17}$$

and

$$\dot{u} = u(p_A - \bar{p}), \tag{18}$$

$$\dot{v} = v(p_B - \bar{p}), \tag{19}$$

$$\dot{w} = w(p_C - \bar{p}). \tag{20}$$

We analyze this system of differential equations under the simplifying assumption of a common cost $c = c_A = c_B = c_C$. The resultant system has fixed points at all three pure strategies, $(u,v,w) = (1,0,0)$, etc., corresponding to a single, self-cooperating species monoculture. In addition, there is an interior fixed point:

$$u^* = \frac{r_B r_C}{r_A r_B + r_A r_C + r_B r_C}, \tag{21}$$

$$v^* = \frac{r_A r_C}{r_A r_B + r_A r_C + r_B r_C}, \tag{22}$$

$$w^* = \frac{r_A r_B}{r_A r_B + r_A r_C + r_B r_C}, \tag{23}$$

which corresponds to a coexistence point of the three species.

To evaluate the stability of this interior fixed point, the eigenvalues of the Jacobian matrix are

$$\zeta_{1,2,3} = \left\{\frac{\theta}{\sigma}, \frac{\tau - \sqrt{3}c\sqrt{\Delta}}{2\sigma^2}, \frac{\tau + \sqrt{3}c\sqrt{\Delta}}{2\sigma^2}\right\}, \tag{24}$$

where

$$\sigma = r_A r_B + r_A r_C + r_B r_C, \tag{25}$$

$$\Delta = -r_A^2 r_B^2 r_C^2 \sigma^2, \tag{26}$$

$$\tau = c r_A r_B r_C \sigma, \tag{27}$$

$$\theta = -c\sigma(2 r_A r_B r_C - \sigma). \tag{28}$$

The fact that $r_A,r_B,r_C > 1$ implies that $\tau > 0$, and thus the interior fixed point is never stable. Furthermore, the fact that Eqs. (18)–(20) are linearly dependent suggests that the first eigenvalue controls the stability of the simplex, which can be proved by simplifying the system to use only u and v with $w = 1 - u - v$. In this case, the eigenvalues of the interior fixed point $u = v = \frac{1}{3}$ are

$$\tilde{\zeta}_{1,2} = \left\{\frac{\tau - \sqrt{3}c\sqrt{\Delta}}{2\sigma^2}, \frac{\tau + \sqrt{3}c\sqrt{\Delta}}{2\sigma^2}\right\}. \tag{29}$$

Thus we have proved the following:

Proposition III.1. When $c_A = c_B = c_C = c$, the three-species public goods game has a unique asymptotically unstable interior fixed point.

Numerically obtained trajectories diverging away from the interior fixed point are shown as phase portraits in Fig. 1, for the symmetric case $r = r_A = r_B = r_C = 1.1$ and $c = 2$, as well as for the off-center case $r_A = 1.8$, $r_B = 2.2$ and $r_C = 1.5$ and $c = 1$. The ultimately dominant species depends sensitively on the initial conditions, although the basins of attraction for the three stable states do not appear to be fractal, as shown in Fig. 2. It is clear from this plot that a small shift near the boundaries between basins will lead to a completely different final state.

A. Minimum growth properties of the interior fixed point

Proposition III.2. Assume $c_A = c_B = c_C = c, r_A,r_B,r_C > 1$ and the following regularity conditions hold:

(1) If $r_A > r_B$, then

$$\frac{r_A r_B}{r_A + r_B} \leqslant r_C \leqslant \frac{r_A r_B}{r_A - r_B}.$$

(2) If $r_A = r_B$, then

$$\frac{r_A r_B}{r_A + r_B} \leqslant r_C.$$

(3) If $r_A < r_B$, then

$$\frac{r_A r_B}{r_A + r_B} \leqslant r_C \leqslant \frac{r_A r_B}{r_B - r_A}.$$

Then the interior fixed point of the three-species public goods game is the point of minimum population growth for the system.

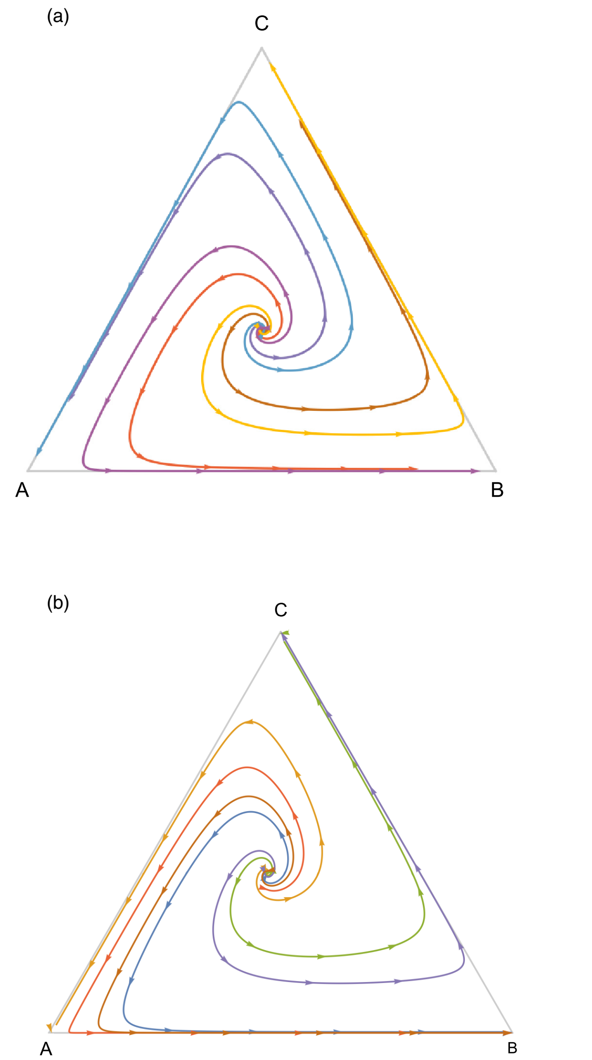

FIG. 1. Phase portraits for the three-species public goods game: (a) with $r_A = r_B = r_C = 1.1$ and $c = 2$, and (b) $r_A = 1.8, r_B = 2.2$ and $r_C = 1.5$ and $c = 1$, which displaces the fixed point off-center.

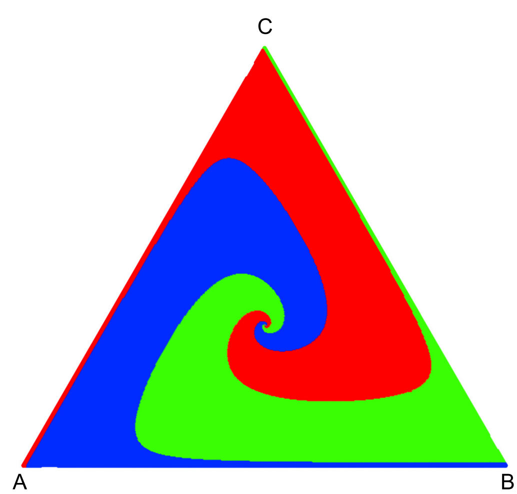

FIG. 2. Basins of attraction for the three-species public good game when $r_A = r_B = r_C = 1.1$ and $c = 2$.

Proof. Consider the optimization problem:

$$\begin{aligned} &\min \; z \\ &s.t. \; p_A(u,v,w) \leqslant z \\ &\qquad p_B(u,v,w) \leqslant z \\ &\qquad p_C(u,v,w) \leqslant z \\ &\qquad u+v+w = 1 \\ &\qquad u,v,w \geqslant 0. \end{aligned} \tag{30}$$

Any solution (u,v,w) represents a species mixture that minimizes the *fastest* species growth, since p_A, p_B, and p_C are growth rates for species A, B, and C, respectively, and we are finding the *minimal* z so that $p_A, p_B, p_C \leqslant z$.

Note the functions p_A, p_B, and p_C are linear in u, v, and w. Consequently, the Karush-Kuhn-Tucker (KKT) conditions are both necessary and sufficient for the solution of problem (30) [27]. (See Appendix A for a summary of relevant results on the KKT optimization conditions.)

We now formulate and solve the specific KKT conditions for problem (30). In doing so, we assume that gradients are taken as $\langle \frac{\partial}{\partial z}, \frac{\partial}{\partial u}, \frac{\partial}{\partial v}, \frac{\partial}{\partial w} \rangle$. Following the formulation in Appendix A, the Kuhn-Tucker equality for Eq. (30) is

$$\begin{aligned} &\nabla z + \lambda_1 \nabla(p_A - z) + \lambda_2 \nabla(p_B - z) + \nabla(p_C - z) \\ &+\mu \nabla(u+v+w-1) + \rho_1 \nabla(-u) + \rho_2 \nabla(-v) \\ &+\rho_3 \nabla(-w) = \mathbf{0}. \end{aligned}$$

The last three terms of the left-hand side are from the constraints $-u \leqslant 0, \; -v \leqslant 0$ and $-w \leqslant 0$. The remainder of the KKT conditions are given by the following:

$$\text{Primal Feasibility} \begin{cases} p_A(u,v,w) - z \leqslant 0 \\ p_B(u,v,w) - z \leqslant 0 \\ p_C(u,v,w) - z \leqslant 0 \\ u+v+w-1 = 0 \\ u,v,w \geqslant 0. \end{cases}$$

$$\text{Dual Feasibility} \begin{cases} \lambda_1,\lambda_2,\lambda_3 \geqslant 0 \\ \rho_1,\rho_2,\rho_3 \geqslant 0 \\ \mu \in \mathbb{R}. \end{cases}$$

$$\text{Complementary Slackness} \begin{cases} \lambda_1(p_A - z) = 0 \\ \lambda_2(p_B - z) = 0 \\ \lambda_3(p_C - z) = 0 \\ \rho_1 u = 0 \\ \rho_2 v = 0 \\ \rho_3 w = 0. \end{cases}$$

For simplicity, assume $u,v,w > 0$ and thus $\rho_1,\rho_2,\rho_3 = 0$, and that $\lambda_1,\lambda_2,\lambda_3 > 0$ and thus $p_A = p_B = p_C = z$. Under the assumption that $c_A = c_B = c_C = c$, the conditions reduce to the following systems:

$$\lambda_1 + \lambda_2 + \lambda_3 = 1, \tag{31}$$

$$c r_A(\lambda_1 + \lambda_2) + \mu = 0, \tag{32}$$

$$cr_B(\lambda_2 + \lambda_3) + \mu = 0, \tag{33}$$

$$cr_C(\lambda_1 + \lambda_3) + \mu = 0, \tag{34}$$

$$cr_A u + cr_C w = c + z, \tag{35}$$

$$cr_A u + cr_B v = c + z, \tag{36}$$

$$cr_B v + cr_C w = c + z, \tag{37}$$

$$u + v + w = 1. \tag{38}$$

These systems are separable and can be solved independently for the primal variables u, v, w, and z and the dual variables λ_1, λ_2, λ_3, and μ. The unique solution is

$$u = \frac{r_B r_C}{\sigma}, \quad v = \frac{r_A r_C}{\sigma},$$
$$w = \frac{r_A r_B}{\sigma},$$
$$z = \frac{2cr_A r_B r_C}{\sigma} - c,$$

and

$$\lambda_1 = \frac{r_A(r_B - r_C) + r_B r_C}{\sigma}, \quad \lambda_2 = \frac{r_B(r_C - r_A) + r_A r_C}{\sigma},$$
$$\lambda_3 = \frac{r_C(r_A - r_B) + r_A r_B}{\sigma}, \quad \mu = -\frac{2cr_A r_B r_C}{\sigma}.$$

Recall σ is defined in Eq. (25). Thus as long as $\lambda_1,\lambda_2,\lambda_3 \geqslant 0$, the interior fixed point is a KKT point (global minimizer) of the linear program in expression (30). Assuming $r_A,r_B,r_C \geqslant 1$, the following regularity conditions,

(1) If $r_A > r_B$, then
$$\frac{r_A r_B}{r_A + r_B} \leqslant r_C \leqslant \frac{r_A r_B}{r_A - r_B},$$

(2) If $r_A = r_B$, then
$$\frac{r_A r_B}{r_A + r_B} \leqslant r_C,$$

(3) If $r_A < r_B$, then
$$\frac{r_A r_B}{r_A + r_B} \leqslant r_C \leqslant \frac{r_A r_B}{r_B - r_A},$$

are equivalent to the statement that $\lambda_1,\lambda_2,\lambda_3 \geqslant 0$ and thus the interior fixed point is a point of minimal population growth. This completes the proof. ■

For the symmetric case when $r_A = r_B = r_C = r$, it is clear the regularity conditions are satisfied (since $r_C = r > r/2$). Thus, the totally mixed population is a point of minimal population growth.

B. Population collapse

Let us now assume that the regularity conditions are met so that the interior fixed point is a point of minimal population growth. In particular, the total population is growing at this point whenever $z > 0$. As in Sec. II, let M denote the entire population. Then $\dot{M} \geqslant 0$ if and only if

$$\frac{2r_A r_B r_C}{r_A r_B + r_A r_C + r_B r_C} - 1 \geqslant 0. \tag{39}$$

In the completely symmetric case $r_A = r_B = r_C = r$, this reduces to $r \geqslant 3/2$. Thus, for rates of return less than this, the population will collapse. Analysis of the complete population leads to further insight regarding the value z in expression (30). Note that in Eq. (3), $\dot{M} = M\bar{p}$. Computing $\bar{p}$ at the interior equilibrium point yields exactly z; thus the growth rate of the total population at this rest point is $\bar{p} = z$, and it becomes clear why the population collapses when $z < 0$. The remaining dual variables $(\lambda_1,\lambda_2,\lambda_3)$ express an alternate, less intuitive relationship between $\bar{p}$ and the amounts that p_A, p_B, and p_C can differ from $\bar{p}$ in an optimal solution.

IV. STABLE COEXISTENCE IN THE CYCLIC PUBLIC GOODS MODEL—PENALTY TAX

Intuitively, the cyclic public goods model we have presented is unstable at its interior fixed point because, as one population gains some proportional advantage and leaves the interior fixed point, that population's payoff begins to increase (if only marginally). The system is then driven by a kind of oscillating predation until only one species is left. For our simple model, a single species population is the only stable outcome.

With the goal of stabilizing coexistence, we modify our model in order to change the stability conditions of the internal fixed point. We introduce an imposed reduction in growth to slow down the dynamics of the increase which drives the system towards a single species, as described above. To accomplish this, we include a proportional penalty or tax on each population, in a modified form of Eqs. (12)–(14):

$$\tilde{p}_A = p_A - \beta_A u, \tag{40}$$

$$\tilde{p}_B = p_B - \beta_B v, \tag{41}$$

$$\tilde{p}_C = p_C - \beta_C w, \tag{42}$$

where β_A, β_B, and β_C are the tax rates per individual levied on each population. Here $\bar{p}$ is the population weighted average payoff with the tax terms β_A, β_B, and β_C included, and we drop the tildes and use the new functions as before. We also assume population dynamics like Eqs. (15)–(17), with tax terms included. Thus while the individuals still reap the benefits of freeloading as well as investment in the public goods games, a penalty tax proportional to their share of the population is also subtracted.

Note that there is no redistribution back to the population of what we are referring to as a tax, nor is it being used to fund any public goods; we are not really addressing here the optimal income tax problem (see, e.g., [28]), though this is clearly not completely unrelated.

Equations (18)–(20) still hold. However, the interior fixed point is now perturbed by the tax rates, so that if $c_A = c_B = c_C = c$ as before and $\beta_A = \beta_B = \beta_C = \beta$ (a flat tax rate across the population), then the interior fixed point (if it exists) is

$$u^* = \frac{\beta^2 + c^2 r_B r_C - \beta c r_B}{3\beta^2 + c^2\sigma - \beta c(r_A + r_B + r_C)}, \tag{43}$$

$$v^* = \frac{\beta^2 + c^2 r_A r_C - \beta c r_C}{3\beta^2 + c^2\sigma - \beta c(r_A + r_B + r_C)}, \tag{44}$$

$$w^* = \frac{\beta^2 + c^2 r_A r_B - \beta c r_A}{3\beta^2 + c^2\sigma - \beta c(r_A + r_B + r_C)}, \tag{45}$$

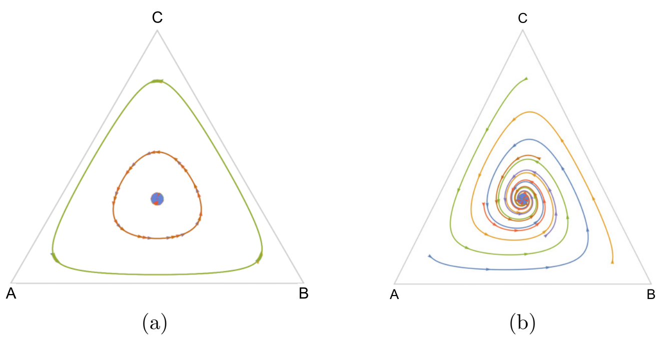

FIG. 3. (a) Center: setting $\beta = 1.5, r_A = r_B = r_C = 3$ and $c_A = c_B = c_C = 1$ results in a neutrally stable nonlinear cycle. (b) Stable: setting $\beta = 2, r_A = r_B = r_C = 3$ and $c_A = c_B = c_C = 1$ results in an asymptotically stable nonlinear cycle.

with σ defined in Eq. (25). Unlike the interior fixed point in the nontaxed case, however, this fixed point can be attracting, neutral, or repelling depending on the relative values of β and r_A, r_B, and r_C. This is illustrated in Fig. 3.

Proposition IV.1. Assume $c_A = c_B = c_C = c$ and

$$cr_A - \beta_A > 0, \quad cr_B - \beta_B > 0, \quad cr_C - \beta_C > 0.$$

Then there is a unique interior fixed point for the three-species public goods game with taxation. Furthermore, let

$$\tilde{\mathbf{A}} = \begin{bmatrix} 0 & \beta_B - cr_B & \beta_C \\ \beta_A & 0 & \beta_C - cr_C \\ \beta_A - cr_A & \beta_B & 0 \end{bmatrix}. \tag{46}$$

Then the following holds:

(1) If $\det \tilde{\mathbf{A}} > 0$, then the interior fixed point is (globally) asymptotically stable.

(2) If $\det \tilde{\mathbf{A}} = 0$, then the interior fixed point is a nonlinear center.

(3) If $\det \tilde{\mathbf{A}} < 0$, then the interior fixed point is asymptotically unstable.

Consequently, the three-species public goods game with taxation does not admit an isolated orbit (limit cycle).

Proof. Note first that

$$\bar{p} = -c + (cr_A - \beta_A)u^2 + cr_C uw + (cr_B - \beta_B)v^2 + cr_A uv + (cr_C - \beta_C)w^2 + cr_B vw. \tag{47}$$

In particular, $p_A - \bar{p}$, $p_B - \bar{p}$, and $p_C - \bar{p}$ have no constant terms (i.e., terms containing only a constant multiple of c). Moreover, let

$$\mathbf{A} = \begin{bmatrix} cr_A - \beta_A & 0 & cr_C \\ cr_A & cr_B - \beta_B & 0 \\ 0 & cr_B & cr_C - \beta_C \end{bmatrix}. \tag{48}$$

We can rewrite the equations governing $\dot{u}$, $\dot{v}$, and $\dot{w}$, Eqs. (18)–(20), using the replicator dynamics given in expression (7) with (payoff) matrix $\mathbf{A}$.

As noted in Sec. II, the dynamics of the system are unchanged when constants are added to the columns of the matrix $\mathbf{A}$. Thus, we construct $\tilde{\mathbf{A}}$ from $\mathbf{A}$ by adding $\beta_A - cr_A$ to the first column, $\beta_B - cr_B$ to the second column, and $\beta_C - cr_C$ to the third column. Under our assumption that $\beta_A - cr_A$, $\beta_B - cr_B$, $\beta_C - cr_C < 0$, this is a rock-paper-scissors matrix and the RPS replicator dynamics are diffeomorphic to the original three-species triple public goods game with taxation. The result follows immediately from Lemma II.1. ■

Setting $\beta_A = \beta_B = \beta_C = 1$, $c = 1$, and $r_A = r_B = r_C = 2$ we recover the standard RPS payoff matrix [22]. Intuitively, public good A maps to *rock*, public good B maps to *paper*, and public good C maps to *scissors*.

A. Totally symmetric case

It is instructive to consider the totally symmetric three-species public goods game. The following corollary is illustrated in Fig. 3.

Corollary IV.2. Assume $c_A = c_B = c_C = c$, $\beta_A = \beta_B = \beta_C = \beta$, and $r_A = r_B = r_C = r$, then:

(1) If $\beta > rc/2$, then the fixed point $u = v = w = \frac{1}{3}$ is globally asymptotically stable.

(2) If $\beta < rc/2$, then the fixed point $u = v = w = \frac{1}{3}$ is asymptotically unstable.

(3) If $\beta = rc/2$, then the fixed point $u = v = w = \frac{1}{3}$ is a neutrally stable nonlinear center.

Proof. In this case,

$$\tilde{\mathbf{A}} = \begin{bmatrix} 0 & \beta - cr & \beta \\ \beta & 0 & \beta - cr \\ \beta - cr & \beta & 0 \end{bmatrix}.$$

Computing the determinant we have

$$\det \tilde{\mathbf{A}} = (2\beta - cr)[\beta^2 + cr(cr - \beta)]. \tag{49}$$

We assumed $cr - \beta > 0$; thus the sign of the determinant is entirely controlled by the sign of $2\beta - cr$. The result follows immediately. ■

The previous corollary asserts precisely what we intuitively believed: adding a tax on population growth that is *high enough* will essentially alter the stability of the mixed population and allow for species coexistence. However, a tax that is too high could again result in population collapse, as we discuss next.

Proposition IV.3. Assume the conditions of Corollary IV.2. The point $u = v = w = \frac{1}{3}$ is still a point of minimal growth in the presence of taxation.

Proof. Accounting for the terms β_A, β_B, and β_C, Eqs. (31)–(38) can be modified as

$$\begin{aligned} \lambda_1 + \lambda_2 + \lambda_3 &= 1, \\ (cr_A - \beta_A)\lambda_1 + cr_A\lambda_2 + \mu &= 0, \\ (cr_B - \beta_B)\lambda_2 + cr_B\lambda_3 + \mu &= 0, \\ (cr_C - \beta_C)\lambda_3 + cr_C\lambda_3 + \mu &= 0, \\ (cr_A - \beta_A)u + cr_C w &= c + z, \\ cr_A u + (cr_B - \beta_B)v &= c + z, \\ cr_B v + (cr_C - \beta_C)w &= c + z, \\ u + v + w &= 1. \end{aligned}$$

When $r_A = r_B = r_C = r$ and $\beta_A = \beta_B = \beta_C = \beta$, these equations have solution $u = v = w = \frac{1}{3}$, $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{3}$, $z = \frac{1}{3}(c(2r - 3) - \beta)$, and $\mu = \frac{2}{3}(\beta - cr)$. This point is a KKT point and must minimize population growth. ■

Corollary IV.4. If

$$\beta > c(2r-3),$$

then the population will collapse. Consequently, since we assume $\beta \geqslant 0$, we again see $r \geqslant 3/2$ is a necessary condition to prevent population collapse.

This result is unexpected and intriguing because it suggests that the minimal growth property of the interior fixed point is not related at all to its stability.

V. OPTIMAL STABILIZING PENALTY TAX

Consider now the problem of driving the population towards the mixed-species state. Such a problem could arise for purposes of minimizing population growth (as shown in Proposition IV.3), or for the purposes of species diversity, or both.

The penalty tax in our model is never redistributed back to the population in any way and as such represents an inefficiency or waste. We therefore seek a minimal tax that drives these public goods populations toward a mixed state. For simplicity we study the problem in the totally symmetric case: $c_A = c_B = c_C = c$, $r_A = r_B = r_C = r$, with u^*, v^*, and w^* the mixed population stationary point that is attracting when $\beta > rc/2$. We can phrase the minimal taxation problem as an optimal control problem:

$$\begin{aligned} \min\ & \Psi[u(t_f),v(t_f),w(t_f)] + \int_0^{t_f} \frac{1}{2}(u-u^*)^2 \\ &+ \frac{1}{2}(v-v^*)^2 + \frac{1}{2}(w-w^*)^2 + \frac{1}{2}k\beta^2\, dt \\ s.t.\ & \dot{u} = u(p_A - \bar{p}) \\ & \dot{v} = v(p_B - \bar{p}) \\ & \dot{w} = w(p_C - \bar{p}) \\ & u(0)=u_0,\ v(0)=v_0,\ w(0)=w_0 \\ & \frac{cr}{2} \leqslant \beta \leqslant c(2r-3). \end{aligned} \tag{50}$$

Here, p_A, p_B, and p_C are functions of u, v, w, and β. In the *closed-loop* setting, β is a function of u, v, and w, while in the *open-loop* setting, it is just a function of t. In either case, β is time varying. We require $\beta \geqslant \frac{cr}{2}$ to ensure the control produces a stable system and $\beta \leqslant c(2r-3)$ to prevent population collapse.

This optimal control problem is highly nonlinear and does not necessarily admit a closed form optimal control. Moreover, the inclusion of boundaries on the control function [i.e., $\frac{cr}{2} \leqslant \beta \leqslant c(2r-3)$] may present difficulties as there may be periods of time when the control function would do better to escape these boundaries but cannot. In such a scenario, the population size might be *reduced* to bring it closer to the equilibrium point.

As an alternative to approaching this problem numerically, we will do the following:

(1) We will quasilinearize the problem around the state variables, leaving a simpler but still tractable nonlinear optimal control problem.

(2) We will define a new control function $\gamma = \beta - \frac{cr}{2}$ with minimum bound at zero, which will be *ensured* by the structure of the objective functional.

(3) We will set $\Psi[u(t_f),v(t_f),w(t_f)] \equiv 0$ and assume that r is *sufficiently large* so that the upper bound can be ignored. Since the new control function $\gamma(t)$ will tend to zero (as a result of the assumption on Ψ), this simplification is justified and will result in an intriguing result.

For convenience and completeness, key facts from optimal control theory used in this approach are included in Appendix B.

Define the matrix and state vector as

$$\mathbf{A}(\beta) = \begin{bmatrix} cr-\beta & 0 & cr \\ cr & cr-\beta & 0 \\ 0 & cr & cr-\beta \end{bmatrix} \quad \mathbf{z} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}. \tag{51}$$

Rewrite the optimal control dynamics as

$$\dot{u} = u[(\mathbf{e}_1 - \mathbf{z})^T \mathbf{A}(\beta)\mathbf{z}], \tag{52}$$

$$\dot{v} = v[(\mathbf{e}_2 - \mathbf{z})^T \mathbf{A}(\beta)\mathbf{z}], \tag{53}$$

$$\dot{w} = w[(\mathbf{e}_3 - \mathbf{z})^T \mathbf{A}(\beta)\mathbf{z}]. \tag{54}$$

Following the approach in [29], let

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{55}$$

Replace β with $\beta = \gamma + \frac{cr}{2}$, thus ensuring that the system is (neutrally) stable whenever $\gamma \geqslant 0$. We can then write

$$\mathbf{A}(\gamma) = cr\mathbf{L} + \left(\gamma + \frac{cr}{2}\right)\mathbf{M} = cr\left(\mathbf{L} + \frac{1}{2}\mathbf{M}\right) + \gamma\mathbf{M}.$$

We next define the following functions:

$$f_0(\mathbf{z},\gamma) = \tfrac{1}{2}||\mathbf{z} - \mathbf{z}^*||^2 + \tfrac{1}{2}k\gamma^2, \tag{56}$$

$$\mathbf{F}_i(\mathbf{z}) = \mathbf{z}_i(\mathbf{e}_i - \mathbf{z})^T \left(\mathbf{L} + \tfrac{1}{2}\mathbf{M}\right)\mathbf{z}, \tag{57}$$

$$\mathbf{G}_i(\mathbf{z}) = \mathbf{z}_i(\mathbf{e}_i - \mathbf{z})^T \mathbf{M}\mathbf{z}. \tag{58}$$

Assuming $\Psi[\mathbf{z}(t_f)] \equiv 0$, the modified optimal control problem is

$$\begin{aligned} \min\ & \int_0^{t_f} f_0(\mathbf{z},\gamma)\, dt \\ s.t.\ & \dot{\mathbf{z}}_i = cr\mathbf{F}(\mathbf{z}) + \gamma\mathbf{G}(\mathbf{z}) \\ & \mathbf{z}(0) = \mathbf{z}_0 \\ & \gamma \in \mathbb{R}. \end{aligned} \tag{59}$$

We further assume that both $\mathbf{z}$ and γ are appropriately square integrable and furthermore will require that γ is time differentiable almost everywhere. The Hamiltonian for this system is then

$$\mathcal{H}(\mathbf{z},\boldsymbol{\lambda},\gamma) = f_0(\mathbf{z},\gamma) + cr\boldsymbol{\lambda}^T\mathbf{F}(\mathbf{z}) + \gamma\boldsymbol{\lambda}^T\mathbf{G}(\mathbf{z}). \tag{60}$$

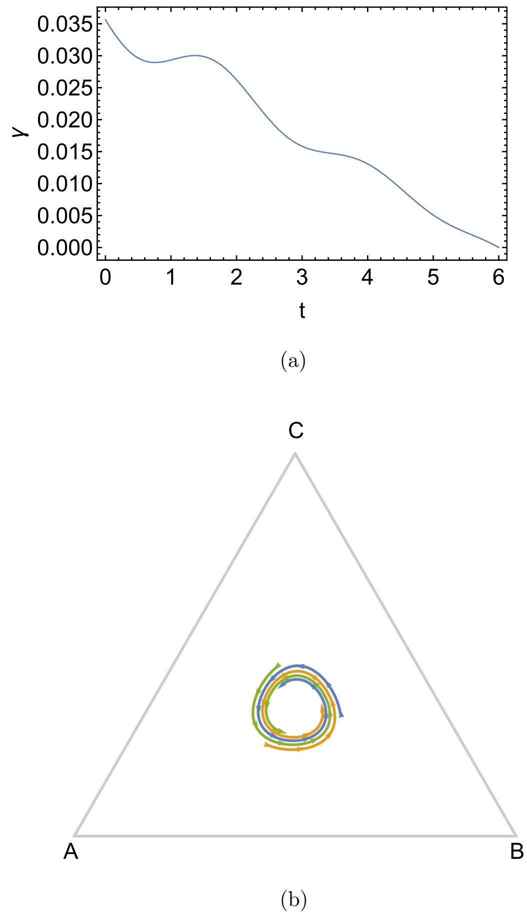

FIG. 4. (a) A computed control function $\gamma(t)$ that satisfies the Euler-Lagrange necessary conditions. Here $t_f = 6$. (b) The state dynamics under the optimal control function with starting values near the fixed point. Here $t_f = 10$.

If the Hamiltonian is convex, the optimal γ should satisfy

$$\gamma^* = -\frac{1}{k}\boldsymbol{\lambda}^T \mathbf{G}(\mathbf{z}). \tag{61}$$

By Lemma B.1, the Euler-Lagrange necessary conditions are

$$\begin{aligned}
dot\mathbf{z} &= cr\mathbf{F}(\mathbf{z}) - \left(\frac{1}{k}\boldsymbol{\lambda}^T\mathbf{G}(\mathbf{z})\right)\mathbf{G}(\mathbf{z})\\
\dot{\boldsymbol{\lambda}}^T &= -(\mathbf{z}-\mathbf{z}^*)^T - cr\boldsymbol{\lambda}^T\frac{\partial \mathbf{F}(\mathbf{z})}{\partial \mathbf{z}}\\
&\quad + \frac{1}{k}\boldsymbol{\lambda}^T\mathbf{G}(\mathbf{z})\boldsymbol{\lambda}^T\frac{\partial \mathbf{G}(\mathbf{z})}{\partial \mathbf{z}}\\
\mathbf{z}(0) &= \mathbf{z}_0\\
\boldsymbol{\lambda}(t_f) &= \mathbf{0}.
\end{aligned}$$

This nonlinear ODE system can be solved numerically to obtain a solution satisfying the necessary conditions for the optimal controller. An example γ^* is shown in Fig. 4(a), while the resulting state dynamics under that control are shown in Fig. 4(b). The convexity of the objective function is not sufficient to ensure the optimality of the control since the resulting Hamiltonian is not necessarily guaranteed to be convex in both the state and control. However, for certain k, we can find a quasilinearized variation of this problem that has a convex Hamiltonian and admits solutions close to the optimal control. Before proceeding, it is worth noting that control problems with this form are considered in [30], where a control Lyapunov method is also discussed, and in [31], where receding horizon control is considered. Neither reference derives a sufficient second order ODE for the control function, as we do next.

The form of the Hamiltonian suggests a reasonable quasilinearization of the equation of motion to be

$$\dot{\mathbf{z}} \approx \left(cr\frac{\partial \mathbf{F}(\mathbf{z})}{\partial \mathbf{z}} + \gamma\frac{\partial \mathbf{G}(\mathbf{z})}{\partial \mathbf{z}}\right)(\mathbf{z}-\mathbf{z}^*), \tag{62}$$

where $\mathbf{z}^*$ is the goal fixed point. We call this quasilinearized because we are linearizing on the state, not the control variable. Without loss of generality, assume we translate the system so that $\mathbf{z}^* = \mathbf{0}$. This assumption will play a critical role in the proof of Lemma V.3. Let

$$\mathbf{J} = cr\frac{\partial \mathbf{F}(\mathbf{z})}{\partial \mathbf{z}}, \qquad \mathbf{H} = \frac{\partial \mathbf{G}(\mathbf{z})}{\partial \mathbf{z}}.$$

Proposition V.1. Assume we have translated so that $\mathbf{z}^* = \mathbf{0}$. If for all $t \in [0,t_f]$, $||\boldsymbol{\lambda}||^2 < 9k$, then

$$\gamma^* = -\frac{1}{k}\boldsymbol{\lambda}^T\mathbf{H}\mathbf{z} \tag{63}$$

is the optimal control for the quasilinearized control problem:

$$\begin{aligned}
\min &\int_0^{t_f} f_0(\mathbf{z},\gamma)\,dt\\
s.t.\ \ \dot{\mathbf{z}}_i &= cr\mathbf{J}\mathbf{z} + \gamma\mathbf{H}\mathbf{z}\\
\mathbf{z}(0) &= \mathbf{z}_0\\
\gamma &\in \mathbb{R}.
\end{aligned} \tag{64}$$

Proof. The quasilinearized Hamiltonian is then

$$\tilde{\mathcal{H}}(\mathbf{z},\boldsymbol{\lambda},\gamma) = f_0(\mathbf{z},\gamma) + \boldsymbol{\lambda}^T\mathbf{J}\mathbf{z} + \gamma\boldsymbol{\lambda}^T\mathbf{H}\mathbf{z}. \tag{65}$$

The Hessian matrix of $\tilde{\mathcal{H}}$ as a function of $\mathbf{z}$ and γ has eigenvalues

$$\begin{aligned}
r_{1,2,3,4} = \Big\{&1,1,\tfrac{1}{6}\big(3+3k-\sqrt{9(k-1)^2+4||\boldsymbol{\lambda}||^2}\big),\\
&\tfrac{1}{6}\big(3+3k+\sqrt{9(k-1)^2+4||\boldsymbol{\lambda}||^2}\big)\Big\}.
\end{aligned}$$

The first, second, and fourth eigenvalues are always positive. The third eigenvalue is positive whenever $||\boldsymbol{\lambda}||^2 < 9k$. Thus, the Hamiltonian is jointly convex in $\mathbf{z}$ and γ (because the Hessian matrix is positive definite) whenever $||\boldsymbol{\lambda}||^2 < 9k$.

Assume the Hamiltonian is jointly convex in $\mathbf{z}$ and γ. The control *minimizing* the Hamiltonian can be computed by differentiation. The resulting minimizing control is

$$\gamma^* = -\frac{1}{k}\boldsymbol{\lambda}^T\mathbf{H}\mathbf{z}.$$

By the restricted Mangaserian sufficiency theorem (Lemma B.2) this is the optimal control when the Hamiltonian is jointly convex in $\mathbf{z}$ and γ, which is implied when $||\boldsymbol{\lambda}||^2 < 9k$. This completes the proof. ■

The resulting Euler-Lagrange necessary (and conditionally sufficient) conditions for the quasilinearized system are thus

$$\dot{\mathbf{z}} = \mathbf{J}\mathbf{z} + \gamma \mathbf{H}\mathbf{z}$$
$$\dot{\boldsymbol{\lambda}}^T = -\mathbf{z}^T - \boldsymbol{\lambda}^T\mathbf{J} - \gamma\boldsymbol{\lambda}^T\mathbf{H}$$
$$\mathbf{z}(0) = \mathbf{z}_0$$
$$\boldsymbol{\lambda}(t_f) = \mathbf{0}.$$

Throughout the remainder of this paper, we will assume that k is sufficiently large so that $||\boldsymbol{\lambda}||^2 < 9k$, though in practice this may need to be verified *a posteriori*.

Lemma V.2. Assume γ is an optimal control for the quasilinearized optimal control problem, expression (64). Then the time derivative of the optimal control obeys

$$\dot{\gamma} = \frac{1}{k}\mathbf{z}^T\mathbf{H}\mathbf{z}, \tag{66}$$

$$\gamma(t_f) = 0. \tag{67}$$

Proof. Differentiate Eq. (63). The result is

$$\dot{\gamma} = -\frac{1}{k}(\dot{\boldsymbol{\lambda}}^T\mathbf{H}\mathbf{z} + \boldsymbol{\lambda}^T\mathbf{H}\dot{\mathbf{z}}).$$

Expanding, we have

$$\dot{\gamma} = \frac{1}{k}\mathbf{z}^T\mathbf{H}\mathbf{z} + \frac{1}{k}(\boldsymbol{\lambda}^T(\mathbf{J} + \gamma\mathbf{H})\mathbf{H}\mathbf{z}) - \frac{1}{k}\boldsymbol{\lambda}^T\mathbf{H}(\mathbf{J}\mathbf{z} + \gamma\mathbf{H}\mathbf{z}).$$

Then we can write

$$\dot{\gamma} = \frac{1}{k}\mathbf{z}^T\mathbf{H}\mathbf{z} + \frac{1}{k}(\boldsymbol{\lambda}^T(\mathbf{J}\mathbf{H} - \mathbf{H}\mathbf{J})\mathbf{z} + \gamma\boldsymbol{\lambda}^T(\mathbf{H}\mathbf{H} - \mathbf{H}\mathbf{H})\mathbf{z})$$
$$= \frac{1}{k}\mathbf{z}^T\mathbf{H}\mathbf{z} + \frac{1}{k}\boldsymbol{\lambda}^T(\mathbf{J}\mathbf{H} - \mathbf{H}\mathbf{J})\mathbf{z}.$$

In this case,

$$\mathbf{J} = cr\cdot\begin{bmatrix} -\frac{1}{6} & -\frac{1}{3} & 0 \\ 0 & -\frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{3} & 0 & -\frac{1}{6} \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} -\frac{1}{9} & \frac{2}{9} & \frac{2}{9} \\ \frac{2}{9} & -\frac{1}{9} & \frac{2}{9} \\ \frac{2}{9} & \frac{2}{9} & -\frac{1}{9} \end{bmatrix},$$

and note that $\mathbf{J}\mathbf{H} = \mathbf{H}\mathbf{J}$. Thus we have proved

$$\dot{\gamma} = \frac{1}{k}\mathbf{z}^T\mathbf{H}\mathbf{z}.$$

By the transversality condition, $\boldsymbol{\lambda}(t_f) = \mathbf{0}$; thus

$$\gamma(t_f) = -\frac{1}{k}\boldsymbol{\lambda}^T(t_f)\mathbf{H}\mathbf{z}(t_f) = 0.$$

This completes the proof. ■

Note, $\dot{\gamma}$ is not dependent on the adjoint variables $\boldsymbol{\lambda}$, but only on the state. This expression is very much in the spirit of Eq. (3) of Ref. [32].

Lemma V.3. Assume C is a constant of integration. Then

$$\mathbf{z}^T\mathbf{z} = \gamma^2 + C, \tag{68}$$

where

$$C = \mathbf{z}_0^T\mathbf{z}_0 - \gamma^2(0).$$

Proof. We have

$$\frac{1}{2}\frac{d}{dt}\mathbf{z}^T\mathbf{z} = \mathbf{z}^T(\mathbf{J}\mathbf{z} + \gamma\mathbf{H}\mathbf{z})$$
$$= \mathbf{z}^T\mathbf{J}\mathbf{z} + \gamma\mathbf{z}^T\mathbf{H}\mathbf{z} = \mathbf{z}^T\mathbf{J}\mathbf{z} + \gamma\dot{\gamma} \tag{69}$$

by Lemma V.2, Eq. (66). Computing $\mathbf{z}^T\mathbf{J}\mathbf{z}$ yields

$$cr[z_1 z_2 z_3]\begin{bmatrix} -\frac{1}{6} & -\frac{1}{3} & 0 \\ 0 & -\frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{3} & 0 & -\frac{1}{6} \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$
$$= -\frac{cr}{6}(z_1 + z_2 + z_3)^2. \tag{70}$$

Recall $\mathbf{z}^*$ has been translated to the origin, so if $u + v + w = 1$ and $u^* = v^* = w^* = \frac{1}{3}$, then $z_1 = u - u^*$, $z_2 = v - v^*$, and $z_3 = w - w^*$. This implies that $z_1 + z_2 + z_3 = 0$, because the linearized form of the problem does not depart from Δ_3. Thus

$$\frac{d}{dt}\mathbf{z}^T\mathbf{z} = 2\gamma\dot{\gamma}. \tag{71}$$

By integrating we obtain

$$\mathbf{z}^T\mathbf{z} = \gamma^2 + C,$$

where C is a constant of integration satisfying

$$C = \mathbf{z}_0^T\mathbf{z}_0 - \gamma^2(0).$$

This completes the proof. ■

Proposition V.4. The open-loop optimal control is a solution to the following second order differential equation:

$$k\ddot{\gamma} - \tfrac{2}{9}\gamma(\gamma^2 + C) = 0 = 0, \tag{72}$$

with the boundary conditions

$$\gamma(t_f) = 0, \quad \gamma'(0) = \frac{1}{k}\mathbf{z}_0^T\mathbf{H}\mathbf{z}_0, \quad C = \mathbf{z}_0^T\mathbf{z}_0 - \gamma(0)^2.$$

Proof. Computing $\ddot{\gamma}$ yields

$$k\ddot{\gamma} = \dot{\mathbf{z}}^T\mathbf{H}\mathbf{z} + \mathbf{z}^T\mathbf{H}\dot{\mathbf{z}}$$
$$= (\mathbf{z}^T\mathbf{J}^T + \gamma\mathbf{z}^T\mathbf{H}^T)\mathbf{H}\mathbf{z} + \mathbf{z}^T\mathbf{H}(\mathbf{J}\mathbf{z} + \gamma\mathbf{H}\mathbf{z})$$
$$= \mathbf{z}^T(\mathbf{J}^T\mathbf{H} + \mathbf{H}\mathbf{J})\mathbf{z} + \gamma\mathbf{z}^T(\mathbf{H}^T\mathbf{H} + \mathbf{H}^2)\mathbf{z}. \tag{73}$$

In this case

$$(\mathbf{H}^T\mathbf{H} + \mathbf{H}^2) = \tfrac{2}{9}\mathbf{I}_3 \tag{74}$$

and

$$\mathbf{z}^T(\mathbf{J}^T\mathbf{H} + \mathbf{H}\mathbf{J})\mathbf{z} = -\frac{cr}{9}\left(z_1^2 + z_2^2 + z_3^2\right) = 0, \tag{75}$$

as before. We can therefore write

$$k\ddot{\gamma} = \tfrac{2}{9}\gamma\mathbf{z}^\mathbf{T}\mathbf{z}.$$

Thus, by Lemma V.3 [Eq. (68)], this implies

$$k\ddot{\gamma} - \tfrac{2}{9}\gamma(\gamma^2 + C) = 0$$

and we have the boundary conditions

$$\gamma(t_f) = 0, \quad \gamma'(0) = \frac{1}{k}\mathbf{z}_0^T\mathbf{H}\mathbf{z}_0, \quad C = \mathbf{z}_0^T\mathbf{z}_0 - \gamma(0)^2.$$

■

An example of the controlled quasilinearized system and $\gamma(t)$ found by solving the quasilinearized Euler-Lagrange equations and the second order ODE are shown in Fig. 5

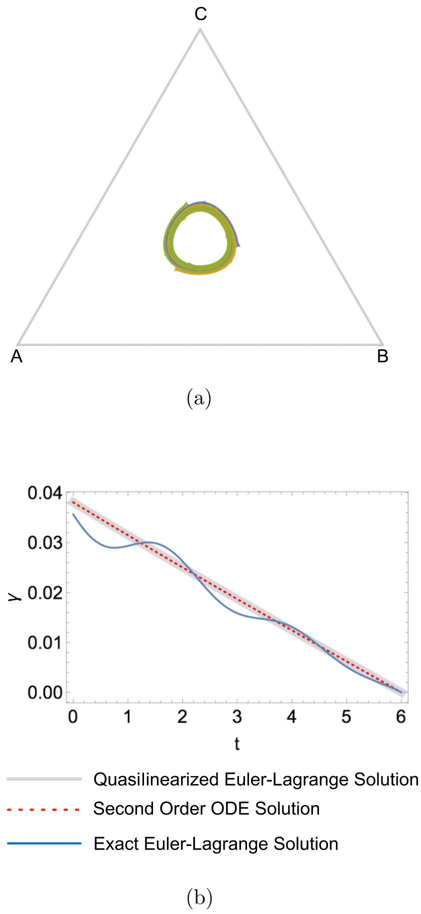

FIG. 5. (a) The state using the quasilinear control function shows slower decay (as expected) than the nonlinear control function. Here $t_f = 20$ to illustrate that the quasilinear controller is still driving the system toward the fixed point. (b) The quasilinear control function is shown and compared to the solution to the second order ODE illustrating their equivalence. We also compare the control function computed for the nonlinear control problem to the quasilinearized control. Here $t_f = 6$.

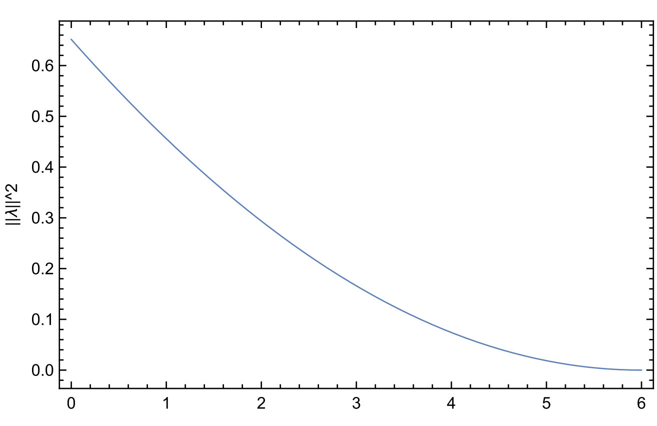

FIG. 6. The plot of $||\boldsymbol{\lambda}||^2$ showing that throughout the entire time its value does not exceed $9k$ ($k = 1$), thus implying that the identified controller is optimal.

(compare to Fig. 4). We also superimpose the optimal control function determined from the nonlinear system. Note the second order ODE is exactly the solution to the Euler-Lagrange equation for $\gamma(t)$. Additionally, we plot $||\boldsymbol{\lambda}||^2$ in Fig. 6, showing that $||\boldsymbol{\lambda}||^2 < 9k$ (here $k = 1$) and thus confirming *a posteriori* that the resulting controls are optimal.

VI. CONCLUSION AND FUTURE DIRECTIONS

In this paper, we proposed a three-species public goods game in which each species both freeloads and contributes to a public good. We showed that when all species pay identical costs, the unique interior mixed population is unstable and the population is ultimately dominated by a single species. We then showed that by imposing a penalty tax in proportion to the size of each population, we could stabilize the interior equilibrium point. We generalized this result by showing that under a common cost assumption, the three-species public goods game with taxation is diffeomorphic to the three-strategy rock-paper-scissors evolutionary game. This immediately implies that the interior fixed point will either be asymptotically stable, unstable, or neutrally stable with a nonlinear center. Consequently, limit cycles cannot form in this system.

In considering the symmetric population case, we assumed that the tax could be controlled exogenously. From this assumption, we formulated an optimal control problem and showed that a quasilinearized form of the problem admitted an optimal control satisfying a specific second order ODE. We compared the quasilinearized optimal control to the fully nonlinear optimal control and illustrated agreement. We also noted a condition on the problem that ensures the Euler-Lagrange necessary conditions are also sufficient.

Future extensions of this work will include generalizations of cyclic public goods games as well as a generalization of the taxation control problem. In particular, our results on control assume a totally symmetric case. Relaxing this assumption will lead to a richer class of control problems. For n species ($n \geqslant 2$), we expect that the public goods game is fully diffeomorphic to generalized Lotka-Volterra dynamics and thus systems with $n \geqslant 4$ may exhibit dramatic dynamics such as chaotic solutions or limit cycles. Confirming this assertion and looking for biologically meaningful instances where this might occur is also an area for future study. In addition, studying assortative interactions, rather than fully mixed interactions, as discussed in [33], may lead to alternative stabilization mechanisms. Finally, although the extremely simple model we have proposed here is not in any way an encapsulation nor a representation of human society, or realistic economic interactions, we should nonetheless underline the stabilizing role played by the penalty tax in terms of coexistence in this evolutionary game.

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APPENDIX A: KARUSH-KUHN-TUCKER CONDITIONS

In Sec. III A we show that the interior fixed point of the dynamical system we consider has a certain minimal growth property. To do this, we make use of the Karush-Kuhn-Tucker (KKT) conditions from the theory of optimization. We state the conditions we require below. Extensive details on KKT conditions can be found in [34–36].

Lemma A.1 (Karush-Kuhn-Tucker theorem). Let $z:\mathbb{R}^n\to\mathbb{R}$ be a differentiable objective function, $g_i:\mathbb{R}^n\to\mathbb{R}$ be differentiable constraint functions for $i=1,\dots,m$, and $h_j:\mathbb{R}^n\to\mathbb{R}$ be differentiable constraint functions for $j=1,\dots,l$. If $\mathbf{x}^*\in\mathbb{R}^n$ is an optimal point satisfying an appropriate regularity condition for the following optimization problem,

$$P\begin{cases}\min z(x_1,\dots,x_n)\\ s.t. g_1(x_1,\dots,x_n)\leqslant 0\\ \quad\vdots\\ g_m(x_1,\dots,x_n)\leqslant 0\\ h_1(x_1,\dots,x_n)=0\\ \quad\vdots\\ h_l(x_1,\dots,x_n)=0,\end{cases}$$

then there exists $\lambda_1,\dots,\lambda_m\in\mathbb{R}$ and $\mu_1,\dots\mu_l\in\mathbb{R}$ so that

$$\text{Primal Feasibility}:\begin{cases}g_i(\mathbf{x}^*)\leqslant 0 & \text{for } i=1,\dots,m\\ h_j(\mathbf{x}^*)=0 & \text{for } j=1,\dots,l\,.\end{cases}$$

$$\text{Dual Feasibility}:\begin{cases}\nabla z(\mathbf{x}^*)+\sum_{i=1}^m\lambda_i\nabla g_i(\mathbf{x}^*)\\ \quad+|\sum_{j=1}^l\mu_j\nabla h_j(\mathbf{x}^*)=\mathbf{0}\\ \lambda_i\geqslant 0 \quad \text{for } i=1,\dots,m\\ \mu_j\in\mathbb{R} \quad \text{for } j=1,\dots,l\,.\end{cases}$$

$$\text{Complementary Slackness}:\begin{cases}\lambda_i g_i(\mathbf{x}^*)=0\\ \quad \text{for } i=1,\dots,m\,.\end{cases}$$

The expression

$$\nabla z(\mathbf{x}^*)+\sum_{i=1}^m\lambda_i\nabla g_i(\mathbf{x}^*)+\sum_{j=1}^l\mu_j\nabla h_j(\mathbf{x}^*)=\mathbf{0}$$

is sometimes called the Kuhn-Tucker equality. Let $\boldsymbol{\lambda}=\langle\lambda_1,\dots,\lambda_m\rangle$ and $\boldsymbol{\mu}=\langle\mu_1,\dots,\mu_l\rangle$. If $z(\mathbf{x})$, $g_1(\mathbf{x}),\dots,g_m(\mathbf{x})$ and $h_1(\mathbf{x}),\dots,h_l(\mathbf{x})$ are *affine*, then any triple $(\mathbf{x},\boldsymbol{\lambda},\boldsymbol{\mu})$ of values satisfying the KKT conditions has the property that $\mathbf{x}$ is a *global minimizer* of $z(\mathbf{x})$ under the given constraints [27].

APPENDIX B: OPTIMAL CONTROL PROBLEMS

In Sec. V we study a problem of stabilizing a mixed population. We present key facts from optimal control theory used in this study. Details are available in [37–39].

A Bolza-type optimal control problem is an optimization problem of the form

$$\begin{aligned}&\min\quad \Psi[\mathbf{x}(t_f)]+\int_{t_0}^{t_f}f[\mathbf{x}(t),\mathbf{u}(t),t]dt\\ &s.t.\ \ \dot{\mathbf{x}}=\mathbf{g}[\mathbf{x}(t),\mathbf{u}(t),t]\\ &\quad\ \mathbf{x}(0)=\mathbf{x}_0.\end{aligned}\tag{B1}$$

When $\Psi[\mathbf{x}(t_f)]\equiv 0$, this is called a Lagrange-type optimal control problem. The vector of variables $\mathbf{x}$ is called the state, while the vector of decision variables $\mathbf{u}$ is called the control. Additional constraints on $\mathbf{u}$, $\mathbf{x}$ or the joint function of $\mathbf{x}$ and $\mathbf{u}$ can be added.

The *Hamiltonian* with adjoint variables (Lagrange multipliers) $\boldsymbol{\lambda}$ for this problem is

$$\mathcal{H}(\mathbf{x},\boldsymbol{\lambda},u)=f[\mathbf{x}(t),\mathbf{u}(t),t]+\boldsymbol{\lambda}^T\mathbf{g}[\mathbf{x}(t),\mathbf{u}(t),t].$$

In what follows, we assume that all $f(\mathbf{x},\mathbf{u},t)$ and $\mathbf{g}(\mathbf{x},\mathbf{u},t)$ are continuous and differentiable in $\mathbf{x}$ and $\mathbf{u}$ and $\Psi[\mathbf{x}(t_f)]$ is continuous and differentiable in $x(t_f)$. A proof of this lemma can be found in almost every book on optimal control (e.g., [39]).

Lemma B.1 (Necessary conditions of optimal control). If $\mathbf{u}^*$ is a solution to optimal control problem (B1), then there is a vector of adjoint variables $\boldsymbol{\lambda}^*$ so that

$$\mathcal{H}[\mathbf{x}^*(t),\mathbf{u}^*(t),\boldsymbol{\lambda}^*(t)]\leqslant[\mathbf{x}^*(t),\mathbf{u}(t),\boldsymbol{\lambda}^*(t)]\tag{B2}$$

for all $t\in[0,T]$ and for all admissible inputs $\mathbf{u}$, and the following conditions hold:

(1) Pontryagin's minimum principle: $\dot{\mathbf{u}}(t)=\frac{\partial\mathcal{H}}{\partial\mathbf{u}}=\mathbf{0}$ and $\frac{\partial^2\mathcal{H}}{\partial\mathbf{u}^2}$ is positive definite,

(2) Co-state dynamics:

$$\dot{\boldsymbol{\lambda}}(t)=-\frac{\partial\mathcal{H}}{\partial\mathbf{x}}=-\boldsymbol{\lambda}^T(t)\frac{\partial\mathbf{g}(\mathbf{x},\mathbf{u})}{\partial\mathbf{x}}+\frac{\partial f(\mathbf{x},\mathbf{u})}{\partial\mathbf{x}},$$

(3) State dynamics: $\dot{x}(t)=\frac{\partial\mathcal{H}}{\partial\boldsymbol{\lambda}}=\mathbf{g}(\mathbf{x},\mathbf{u})$,

(4) Initial condition: $\mathbf{x}(0)=\mathbf{x}_0$, and

(5) Transversality condition: $\boldsymbol{\lambda}(t_f)=\frac{\partial\Psi}{\partial\mathbf{x}}[\mathbf{x}(t_f)]$.

We will use the following restricted form of Mangasarian's sufficiency condition [37] to argue a controller we derive in Sec. V is the optimal controller.

Lemma B.2 (Mangasarian's sufficiency condition – Restricted form). Suppose $(\mathbf{x}^*,\mathbf{u}^*)$ satisfies the necessary conditions from Lemma B.1 and $\mathcal{H}$ is jointly convex in $\mathbf{x}$ and $\mathbf{u}$ for all time and $\Psi[\mathbf{x}(t_f)]\equiv 0$. Then $(\mathbf{x}^*,\mathbf{u}^*)$ is a globally optimal control in the sense that it minimizes the objective functional.

[1] W. F. Lloyd, *Two Lectures on the Checks to Population* (Oxford University, printed by Collingwood, 1833).
[2] G. Hardin, Science **162**, 1243 (1968).
[3] D. Fudenberg and J. Tirole, *Game Theory* (The MIT Press, Cambridge, MA, 1991).
[4] S. J. Brams, *Game Theory and Politics* (Dover Press, New York, 2004).
[5] S. A. Levin, Proc. Natl. Acad. Sci. U. S. A. **111**, 10838 (2014).
[6] J. O. Ledyard, in *The Handbook of Experimental Economics*, edited by J. Kagel and A. Roth (Princeton University Press, Princeton, NJ, 1995), pp. 111–194.
[7] C. Hauert, M. Holmes, and M. Doebeli, Proc. R. Soc. London, Ser. B **273**, 2565 (2006).
[8] C. Hauert, F. Michor, M. Nowak, and M. Doebeli, J. Theor. Biol. **239**, 195 (2006).
[9] M. Archetti and I. Scheuring, J. Theor. Biol. **299**, 9 (2012).
[10] C. Hauert, M. Holmes, and M. Doebeli, Proc. R. Soc. London, Ser. B **273**, 3131 (2006).
[11] C. Hauert, J. Yuichiro, and M. Doebeli, Theor. Popul. Biol. **73**, 257 (2008).
[12] J. Wang, B. Wu, X. Chen, and L. Wang, Phys. Rev. E **81**, 056103 (2010).
[13] M. Archetti and I. Scheuring, Evolution **65**, 1140 (2010).
[14] C. Hauert, A. Traulsen, H. D. Silva, M. A. Nowak, and K. Sigmund, Biol. Theory **3**, 114 (2008).
[15] C. Hauert, J. Theor. Biol. **267**, 22 (2010).
[16] R. Cressman, J.-W. Song, B.-Y. Zhang, and Y. Tao, J. Theor. Biol. **299**, 144 (2012).
[17] J. Y. Wakano, J. Theor. Biol. **247**, 616 (2007).
[18] J. Y. Wakano, M. Nowak, and C. Hauert, Proc. Natl. Acad. Sci. U. S. A. **106**, 7910 (2009).
[19] J. Y. Wakano and C. Hauert, J. Theor. Biol. **268**, 30 (2011).
[20] X.-B. Cao, W.-B. Du, and Z.-H. Rong, Phys. A (Amsterdam, Neth.) **389**, 1273 (2010).
[21] J. Hofbauer and K. Sigmund, Bull. Amer. Math. Soc. **40**, 479 (2003).
[22] J. W. Weibull, *Evolutionary Game Theory* (MIT Press, Cambridge, MA, 1997).
[23] J. Hofbauer and K. Sigmund, *Evolutionary Games and Population Dynamics* (Cambridge University Press, Cambridge, UK, 1998).
[24] G. B. Ermentrout, C. Griffin, and A. Belmonte, Phys. Rev. E **93**, 032138 (2016).
[25] E. C. Zeeman, in *Global Theory of Dynamical Systems*, Lecture Notes in Mathematics No. 819 (Springer, New York, 1980).
[26] J. Hofbauer, Nonlinear Anal. Theory Methods Appl. **5**, 1003 (1981).
[27] M. S. Bazaraa, J. J. Jarvis, and H. D. Sherali, *Linear Programming and Network Flows* (Wiley-Interscience, Hoboken, NJ, 2004).
[28] E. Saez, Rev. Econ. Studies **68**, 205 (2001).
[29] W. B. Cai, Honors Thesis in Mathematics, The Pennsylvania State University, 2014.
[30] Y. Lin and E. D. Sontag, Syst. Contr. Lett. **16**, 393 (1991).
[31] J. A. Primbs, V. Nevistić, and J. C. Doyle, Asian J. Contr. **1**, 14 (1999).
[32] J. Baillieul, J. Optim. Theory Appl. **25**, 519 (1978).
[33] J. Tanimoto, J. Stat. Mech.: Theory Exp. (2014) P05003.
[34] M. S. Bazaraa, H. D. Sherali, and C. M. Shetty, *Nonlinear Programming: Theory and Algorithms* (John Wiley and Sons, New York, 2006).
[35] D. P. Bertsekas, *Nonlinear Programming*, 2nd ed. (Athena Scientific, Belmont, MA, 1999).
[36] M. Avriel, *Nonlinear Programming: Analysis and Methods* (Dover Press, New York, 2003).
[37] O. L. Mangasarian, SIAM J. Control **4**, 139 (1966).
[38] D. E. Kirk, *Optimal Control Theory: An Introduction* (Dover Press, New York, 2004).
[39] T. Friesz, *Dynamic Optimization and Differential Games* (Springer, New York, 2010).