CHAPTER 4

KNOT DYNAMICS IN A DRIVEN HANGING CHAIN: EXPERIMENTAL RESULTS

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Knotted segments are a natural occurence in linear chains subjected to fluctuations. We study experimentally the size and dynamics of knots in a shaken metal chain. For a given set of conditions, we find that the time it takes a knot to slip off the end of the chain depends sensitively on its topology.

1. Introduction

The topological study of knots has in recent years received many new ideas from the *physical* study of knots, or from physical considerations of ideal knots, such as thickness or crossings in projection; this volume contains several chapters on progress made in these directions (for overviews see e.g.[1,2]). In this chapter, we will discuss the results of a physical experiment with a shaken hanging chain. At certain frequencies, spontaneous knots arise. The study of the dynamics of knots with our chain experiment may lead to new considerations in the theory of knots.

Practically all strings or chains, subjected to complicated motions, will end up knotted - from dangling ropes on sailboats to hair to extension cords to DNA[3,4] and other polymers[5]. The competing effects which lead to spontaneous knots are the random exploration of space performed by a fluctuating chain versus the excluded-volume effect - the chain cannot pass through itself. In describing the physics of a real knot, including its ability to bind, one must consider the detailed distribution of tension in the knot, relative to the self-contact points[6,7]. Thus both tension properties and knot topology contribute to strength, corresponding to the well-known fact that some knots hold much better than others. We do not consider here the role

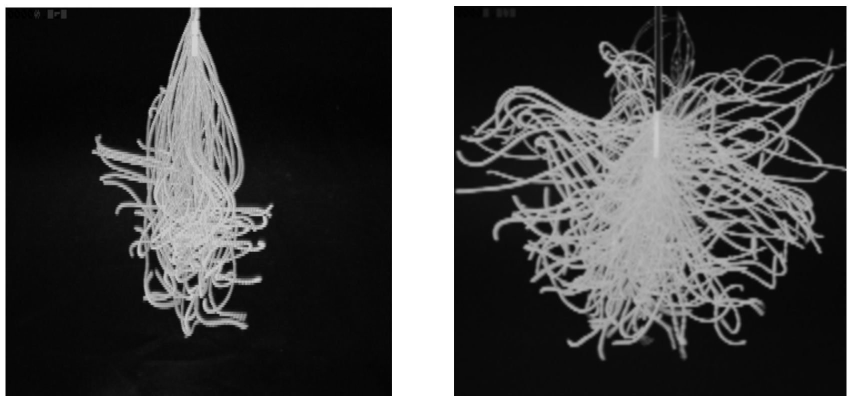

Fig. 1. Experimental collage of the position of a shaken hanging chain at successive times, for $L = 22.7$ cm at two different frequencies (side view). About 8 seconds of data are shown, with $\Delta t = 1/30$ s.

of fluctuations[8], or of charge (long range self-repulsion)[9], in tightening or loosening knots.

2. Our Experimental System

Our experimental study of knots was the accidental result of another study, on the dynamics and patterns of a periodically shaken hanging chain[10]. We used a thin stainless steel chain made of beads of diameter $D = 2.4$ mm, maximum inter-bead spacing 3.4 mm, and linear mass density $\rho = 0.095$ g/cm. The setup is a simple wall-mounted bracket which holds a stepper motor and linear translation stage. The chain is driven sinusoidally by vertically oscillating its upper end via a wheel attached to the translation stage and to the motor, at frequencies f from 1 to 4 Hz and amplitudes $A = 1.3$ cm, 2.0 cm, and 2.7 cm. For more details see Belmonte *et al* [10].

In that experimental study we found several transitions between distinct dynamic states, varying the driving frequency f at fixed amplitude A, with chain lengths L ranging from 7 cm to 80 cm. At certain frequencies we observe a wildly energetic motion (Fig. 1), during which the chain explores its surrounding space, occasionally colliding with itself. For different L or A these same transitions are seen at different frequencies. By introducing the nondimensional amplitude $\varepsilon \equiv A/L$ and frequency $\omega \equiv 2\pi f\sqrt{L/g}$, the transition frequencies collapse onto continuous curves in (ε, ω) parameter space, as shown in Fig. 2. These boundaries were shown by the linear stabil-

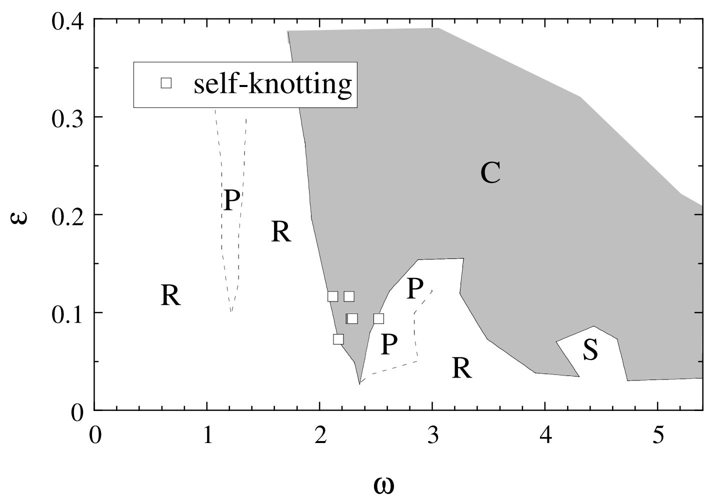

Fig. 2. Points in the (ϵ, ω) parameter space of the hanging chain for which self-knotting (spontaneous knots) have been observed. The letters label the dynamic states of the chain[10]: rod-like (R), pendular (P), stellate (S), and complex (C).

ity analysis of a vertical hanging chain to correspond to Arnol'd tongues[10].

Note that the same metal chain was used independently at Los Alamos to study the dynamics of knots under slightly different conditions: the chain was bounced on a vibrating plate[11,12].

2.1. *Spontaneous Knots*

At certain frequencies in the region of parameter space labelled C for 'complex' in Fig. 2, we noticed previously that the seemingly random motion of the chain could lead to a knot[10]. This is apparently well-known to people who perform certain activities involving long ropes or cords (sailors, rock concert roadees, climbers). The knots vary in complexity, though an early puzzle was that the most frequently observed knot was the figure-eight, and not the simpler trefoil[10]; this is surprising because a simple assumption of random chain motion predicts that knots with fewer crossings should occur more often. The apparent rarity of spontaneous trefoils was explained by the observation that, if a trefoil was tied by hand into the chain, it would rapidly slip off the free end[10].

3. How Long is a Knot?

Although physical knots have been used by humans for centuries, from a mathematical viewpoint the earliest considerations of knots were topological. In this setting the knot is comprised of a knotted curve embedded in $\mathbb{R}^3$; the curve is zero-dimensional (it has no width). Topologically, all representations of a given knot are equivalent, i.e. all are equally valid. Thus it does not matter whether the knot is tight or loose. However, with the beginnings of interest in distinguishing different configurations or representations of the same knot, such as symmetric, relaxed, or with respect to some minimized function such as total curvature[2], more physical considerations were introduced. One could say that the study of knots has expanded from pure topology to physics[1].

A particularly physical concept is the "length" or "ropelength" of a knot; overviews of the history of this idea can be found in[1,2,13]. The basic question is: how much length ΔL of the curve is taken up in a knot? This is a meaningless question without some width or thickness to the curve in which the knot has been tied. Much work has already been done on closed knots (that is, standard topological knots), both numerical and analytic. Recently, these studies have been extended to open knots, which are more often seen in the physical world. Our study involves the dynamics of open knots, including both tying and untying, as well as what we focus on in this chapter: the motion of the knot along the chain.

3.1. *Previous Work*

Pierański *et al.* used the numerical SONO ('shrink on no overlap') algorithm[14] to find 'minimal energy' configurations of several open knots[15]. These knots can be thought of as being in some specifically defined 'tightest' configuration, which results in a certain amount of total length ΔL in the knot itself - the 'ideal length', or knot displacement length. In Fig. 3 we compare previously tabulated values of this ideal length for open and closed knots, both computational[15] and experimental (tied with climbing rope)[16]. Note that we have adjusted the normalization factor for the data in Diao *et al.* to be the diameter and not the radius[16], which gives an extra factor of 2.

3.2. *Open Knot Length in Our Chain*

Using the metal chain from our experiments described above[10], we measured the displacement length ΔL for several open knots. An illustration of

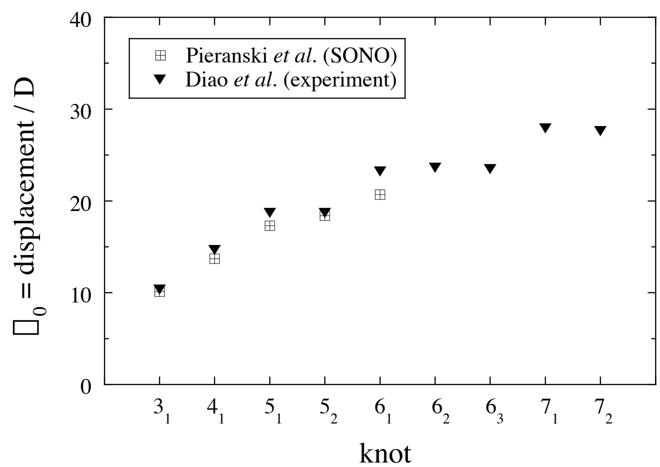

Fig. 3. Comparison of the knot displacement length for open knots seen numerically (with the SONO algorithm)[15] and experimentally[16], as a function of knot type.

this displacement was given in Pierański *et al.*[15] In our system, the knots were tied by hand, and jostled lightly until they were compact; no attempt was made to tighten them by pulling. The reason was that we wanted to obtain a knot size which was appropriate to the dynamic processes experienced by a knot in the shaking chain, in its 'native configuration'. The experimental values of ΔL for several knots in our experimental chain are listed in Table 1.

The open circles in Figure 4 represent the normalization $\Lambda_0 = \Delta L/D$, using $D = 2.4$ mm for the diameter of our chain. However, it is clear from examining these knots that the diameter is not playing the same role as it does in the SONO simulations of ideal knots. This is because our knots are *stiff*; the bending stiffness of the chain provides a hard lower limit to the radius of curvature allowed in the knot configuration. In other words, there is a second lengthscale, the minimum bending radius R of the chain itself, which becomes important because $R > D/2$. Here the diameter plays the same limiting role for the SONO knots, because volume exclusion effectively

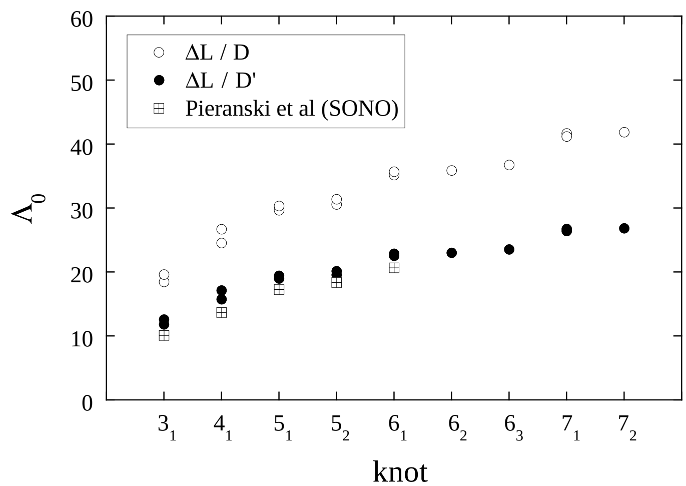

Fig. 4. The normalized displacement length Λ_0 of open knots as measured in our chain, compared to numerical results using the SONO algorithm[15].

defines a bending stiffness for the knot. We will therefore define a knot to be a 'stiff knot' if it is tied in a rope for which $R > D/2$. In this case, the minimum distance across a loop in the knot is either the diameter D or the radius of curvature to centerline of chain: $D' = R - (D/2)$. The difference is determined by whether or not the point of closest approach is in the local plane of curvature. Roughly speaking, the increased stiffness acts as a larger diameter in the plane of curvature (for some of the crossings). For

Table 1. Data on knots in our chain presented in this chapter.

Open knot	ΔL ± 1.5 [mm]	T_u [s]
3_1	44.6	14.0 ± 6.6
4_1	60.0	66.5 ± 1.3
5_1	70.3	12.6 ± 1.2
5_2	72.5	55.2 ± 21.4
6_1	82.9	241.5 ± 89.0
6_2	84.0	119.8 ± 30.0
6_3	86.0	-
7_1	97.1	22.1 ± 10.2
7_2	98.0	-

T_u: for knot tied 10 cm from bottom, with chain $L = 120$ cm, shaken at $f = 3.6$ Hz.

our chain we have $D' = 3.65$ mm, and using this lengthscale we redefine the knot displacement length as $\Lambda'_0 = \Delta L/D'$. We plot our experimentally measured Λ'_0 in Figure 4, and find that it is in much better agreement with the displacement length for open SONO knots[15]. This suggests that the knots we observe in our chain are close to an 'ideal' configuration as defined for *open stiff knots*.

The consideration of stiff knots, like the consideration of the role of fluctuations, brings interesting physical effects into the theory of knots. In each case more room is introduced into the knot configuration, which should affect quantities such as the ropelength. Note that for our chain, the "flexibility" f as defined elsewhere[17] is approximately

$$f = \frac{\kappa}{2/D} \simeq 0.3;$$

by definition $f = 1$ for a perfectly flexible rope[17], thus our knots are indeed rather stiff.

4. Untying Dynamics: Dependence on Knot Type

It was observed initially that a trefoil knot (3_1) would slip off in our chain experiment, whereas a 4_1 knot did not appear to move: when tied 10 cm from the free end of a 28 cm chain, a 3_1 takes about 10 seconds[10]. This observation helped to explain the apparent rarity of spontaneous trefoils in the experiment. It was only later, when different frequencies were tried more systematically, that we observed that a 4_1 would *also* slip off the chain, albeit not nearly as quickly as a 3_1. This opened up the study of the dependence of the slipping motion on knot type.

To test the dependence of slipping motion on knot type, we fixed all other experimental parameters, such as the driving frequency (3.6 Hz), amplitude (2.7 cm), and chain length (120 cm), and tied various knots at a fixed distance from the end of the chain. The time T_u for the knot to untie was measured for knots tied 10 cm from the free end of the chain. As expected from our previous study[10], the 4_1 knot took longer to untie than the 3_1. However, we were surprised to find that there is a strong and complicated dependence of T_u on knot type, ranging from an average of 14 s for the 3_1 to greater than 5 minutes for the 6_3 - in fact, we were not able to measure T_u for this knot.

The average values for T_u as a function of knot type are listed in Table 1, and plotted in Fig. 5. The dependence of T_u on knot type appears quite complicated; note additionally that $T_u(9_1) \simeq 140$ s. For each knot at least

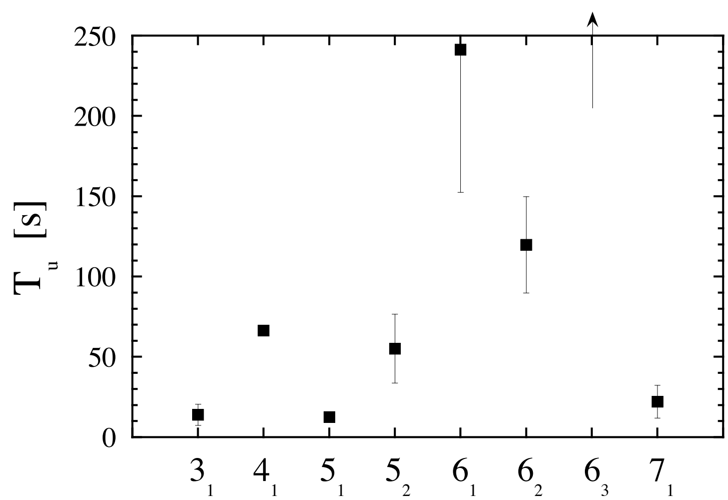

Fig. 5. Dependence of the untying time on knot type, shown for an initial starting length of 10 cm from the bottom (chain with $L = 120$ cm, $f = 3.6$ Hz); the error bars indicate the standard deviation in the distribution of times for the given knot. The arrow represents the fact that the 6_3 knot did not untie during the course of the experiment.

four trials were made, and the individual times measured varied widely. The standard deviations are also reported in Table 1 (as ±) and are plotted as error bars in Fig. 5. This wide variation in untying time suggests a stochastic process, which is currently being modeled. It is not clear to us why some knots display more variance in their untying times than others.

Overall, we do not currently understand the role of topology in determining the apparent speed of a knot on the shaking chain. However, it is interesting to compare the order of knot type from fastest to slowest: $3_1, 5_1, 7_1, 5_2, 4_1, 6_2, 6_1$, with the ordering of 'knot strength' measured by Pieranski *et al* [18], running from easiest to break to hardest, based on the tension required: $3_1, 5_1, 5_2, 7_1, 6_1, 6_2, 4_1$.

Following a suggestion made after a seminar at the University of Pennsylvania in October 2002 (A. Ulyanov, private communication), we found experimentally that the torus knots fall relatively faster than the other

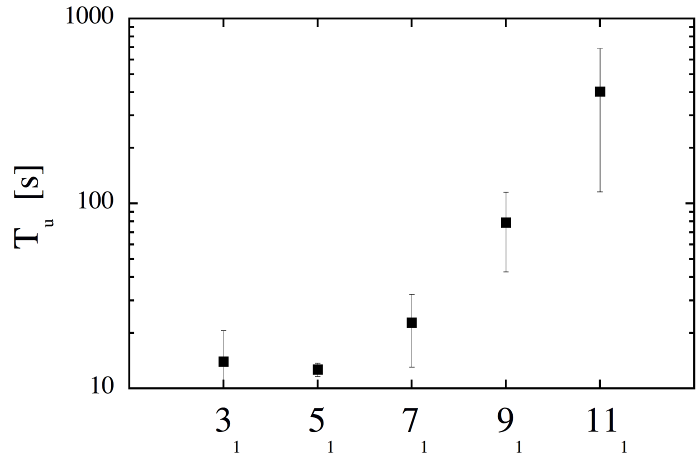

Fig. 6. Dependence of the untying time on knot type for torus knots, under the same conditions as in Fig. 5.

knots. By restricting the plot to only torus knots, as shown in Fig. 6, we at least find an approximately increasing function. A study focusing only on the dynamics of the motion of torus knots and other knot families (such as the twist knots) is currently in progress.

5. Conclusions

We have presented here a new experimental system in which certain physical aspects of knots can be studied, specifically the dynamics of tying and untying, and the slipping motion of a knot along a fluctuating chain. Surprisingly, the slipping motion is very sensitive to topological aspects of the knots, with torus knots being relatively the most efficient at slipping on the chain. However the exact topological quantity which couples to the knot motion is not yet known. This experimental study raises interesting new questions on the coupling of physical properties and topology in knots, for which much remains to be understood.

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